

Landweber-type operator and its properties

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Abstract

Our aim is to present several properties of a Landweber operator and of a Landweber-type operator. These operators are widely used in methods for solving the split feasibility problem and the split common fixed point problem. The presented properties can be used in proofs of convergence of related algorithms.

1 Preliminaries

In this section we recall some notions and facts which we will use in the further part of the paper. Let \mathcal{H} be a real Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and with the corresponding norm $\| \cdot \|$. We say that an operator $S : \mathcal{H} \rightarrow \mathcal{H}$ is *nonexpansive* (NE) if for all $x, y \in \mathcal{H}$ it holds $\|Sx - Sy\| \leq \|x - y\|$. We say that S is α -*averaged*, where $\alpha \in (0, 1)$, if $S = (1 - \alpha)\text{Id} + \alpha N$ for a nonexpansive operator N . We say that S is *firmly nonexpansive* (FNE) if for all $x, y \in \mathcal{H}$ it holds

$$\langle Sx - Sy, x - y \rangle \geq \|Sx - Sy\|^2. \quad (1)$$

By Id we denote the *identity* operator. It follows from (1) that S is FNE if and only if $\text{Id} - S$ is FNE. Denote by $S_\lambda : \text{Id} + \lambda(S - \text{Id})$ a λ -*relaxation* of S , where $\lambda \geq 0$. S is FNE if and only if S_λ is NE for all $\lambda \in [0, 2]$. Moreover, S is a λ -relaxation of an FNE operator if and only if S is $\frac{\lambda}{2}$ -averaged, $\lambda \in (0, 2)$. Let $C \subseteq \mathcal{H}$ be a nonempty closed convex subset. Then for any $x \in \mathcal{H}$ there is a unique $y \in C$ satisfying $\|y - x\| \leq \|z - x\|$ for all $z \in C$. This point is denoted by $P_C x$ and is called the *metric projection* of x onto C . The metric projection P_C is an FNE operator. By $\text{Fix } S := \{z \in \mathcal{H} : Sx = x\}$ we denote the subset of *fixed points* of S . We say that an operator $S : \mathcal{H} \rightarrow \mathcal{H}$ having a fixed point is α -*strongly quasi-nonexpansive* (α -SQNE), where $\alpha \geq 0$, if for all $x \in \mathcal{H}$ and all $z \in \text{Fix } S$ it holds

$$\|Sx - z\|^2 \leq \|x - z\|^2 - \alpha \|Sx - x\|^2. \quad (2)$$

If $\alpha > 0$, then we call an α -SQNE operator *strongly quasi-nonexpansive* (SQNE). A 0-SQNE operator is called *quasi-nonexpansive* (QNE). A λ -relaxation of an FNE operator having a fixed point is $\frac{2-\lambda}{\lambda}$ -SQNE, $\lambda \in (0, 2]$. We say that S is a *cutter* if

$$\langle x - Sx, z - Sx \rangle \leq 0 \quad (3)$$

for all $x \in \mathcal{H}$ and all $z \in \text{Fix } S$. An operator $U : \mathcal{H} \rightarrow \mathcal{H}$ is α -SQNE, if and only if

$$\lambda(Ux - x, z - x) \geq \|Ux - x\|^2 \quad (4)$$

for all $x \in \mathcal{H}$ and all $z \in \text{Fix } U$, where $\lambda = \frac{2}{\alpha+1}$. A FNE operator having a fixed point is a cutter. An operator S is a cutter if and only if S_λ is $\frac{2-\lambda}{\lambda}$ -SQNE, $\lambda \in (0, 2]$. An extended collection of properties of FNE as well as SQNE operators can be found, e.g., in [Ceg12, Chapter 2].

2 Landweber-type operator

Let $\mathcal{H}_1, \mathcal{H}_2$ be real Hilbert spaces, $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator with $\|A\| > 0$, $C \subseteq \mathcal{H}_1$ and $Q \subseteq \mathcal{H}_2$ be nonempty closed convex subsets.

Definition 1 ([Byr02], [Ceg14]) An operator $V : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ defined by

$$V := \text{Id} + \frac{1}{\|A\|^2} A^*(P_Q - \text{Id})A \quad (5)$$

is called a *Landweber operator*. An operator $U : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ defined by

$$U := P_C(\text{Id} + \frac{1}{\|A\|^2} A^*(P_Q - \text{Id})A) \quad (6)$$

is called a *projected Landweber operator*. An operator $R_\lambda : \mathcal{H}_1 \rightarrow \mathcal{H}_1$

$$R_\lambda := P_C(\text{Id} + \frac{\lambda}{\|A\|^2} A^*(P_Q - \text{Id})A), \quad (7)$$

where $\lambda \in (0, 2)$, is called a *projected relaxation* of a Landweber operator V .

A Landweber operator was applied by Landweber [Lan51] in a method for approximating least-squares solution of a first kind integral equation. Censor and Elfving [CE94] introduced the following problem:

$$\text{find } x^* \in C \text{ with } Ax^* \in Q \quad (8)$$

and called it the *split feasibility problem* (SFP). Byrne [Byr02] applied a projected relaxation of a Landweber operator for solving the SFP. Now we introduce a more general operator than the defined by (5).

Definition 2 Let $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ be quasi-nonexpansive. An operator $V : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ defined by

$$V := \text{Id} + \frac{1}{\|A\|^2} A^*(T - \text{Id})A \quad (9)$$

is called a *Landweber-type operator* (related to T).

Because P_Q is quasi-nonexpansive, (5) is a special case of (9). The operators defined by (5), (7) and (9) were applied by many authors for solving the split feasibility problem, the multiple split feasibility problem and the split common fixed point problem (see [Byr02, Ceg14, CS09, Mou10, LMWX12, WX11, Xu10] and the references therein).

3 Properties of a Landweber-type operator

Let $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator with $\|A\| > 0$ and $Q \subseteq \mathcal{H}_2$ be closed convex. By $\text{im } A$ we denote the image of A . The Landweber operator V is closely related to a proximity function $f : \mathcal{H}_1 \rightarrow \mathbb{R}_+$ defined by the following equality

$$f(x) = \frac{1}{2} \|P_Q(Ax) - Ax\|^2. \quad (10)$$

This relation is expressed in the following result, which proof can be found in [Byr02, Proposition 2.1] or [Ceg12, Lemma 4.6.2].

Proposition 3 *The proximity function $f : \mathcal{H}_1 \rightarrow \mathbb{R}$ defined by (10) is a differentiable convex function and $Df = A^*(Ax - P_Q(Ax))$. Moreover, $\text{Fix } V = \text{Argmin}_{x \in \mathcal{H}_1} f$, where $V := \text{Id} + \frac{1}{\|A\|^2} A^*(P_Q - \text{Id})A$ denotes the Landweber operator.*

A sequence $\{U^k x\}_{k=0}^\infty$, where $x \in \mathcal{H}$ is called an *orbit* of an operator $U : \mathcal{H} \rightarrow \mathcal{H}$. Byrne [Byr02, Theorem 2.1] proved that any orbit of the a projected relaxation R_λ of a Landweber operator converges to a solution of the SFP in the case \mathcal{H}_1 and \mathcal{H}_2 are Euclidean spaces, $\lambda \in (0, 2)$. Xu [Xu06] observed that any orbit of R_λ converges weakly in the infinite dimensional case.

Below we give an important property of a Landweber-type operator.

Proposition 4 *If $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ is firmly nonexpansive, then a Landweber-type operator defined by (9) is firmly nonexpansive.*

Proof. (cf. [Ceg12, Theorem 4.6.3], where the case $T = P_Q$ was proved) Recall that U is FNE if and only if $\text{Id} - U$ is FNE. Suppose now that T is FNE, i.e.,

$$\langle (u - Tu) - (v - Tv), u - v \rangle \geq \|(u - Tu) - (v - Tv)\|^2 \quad (11)$$

for all $u, v \in \mathcal{H}_1$. Let $G := \text{Id} - V = \frac{1}{\|A\|^2} A^*(\text{Id} - T)A$, where $V : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ is a Landweber-type operator given by (9). We prove that G is FNE. If we take $u := Ax$ and $v := Ay$ for $x, y \in \mathcal{H}_1$ in inequality (11) and apply the inequality $\|A^*z\| \leq \|A^*\| \cdot \|z\|$ and the equality $\|A^*\| = \|A\|$, then we obtain

$$\begin{aligned} \langle G(x) - G(y), x - y \rangle &= \|A\|^{-2} \langle A^*(Ax - TA x) - A^*(Ay - TA y), x - y \rangle \\ &= \|A\|^{-2} \langle (Ax - TA x) - (Ay - TA y), Ax - Ay \rangle \\ &\geq \|A\|^{-2} \|(Ax - TA x) - (Ay - TA y)\|^2 \\ &= \|A\|^{-4} \|A^*\|^2 \|(Ax - TA x) - (Ay - TA y)\|^2 \\ &\geq \|A\|^{-4} \|A^*(Ax - TA x) - A^*(Ay - TA y)\|^2 \\ &= \|\|A\|^{-2} A^*(Ax - TA x) - \|A\|^{-2} A^*(Ay - TA y)\|^2 \\ &= \|G(x) - G(y)\|^2, \end{aligned}$$

i.e., G is FNE. This yields that a Landweber-type operator $V = \text{Id} - G$ is FNE. ■

Corollary 5 *If $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ is nonexpansive, then a Landweber-type operator V defined by (9) is nonexpansive.*

Proof. The Corollary follows easily from Proposition 4 by an application the fact that S is FNE if and only if $2S - \text{Id}$ is NE. ■

Definition 6 We say that an operator $S : \mathcal{H} \rightarrow \mathcal{H}$ is *asymptotically regular* (AR) if

$$\lim_{k \rightarrow \infty} \|S^{k+1}x - S^kx\| = 0$$

for all $x \in \mathcal{H}$.

Asymptotically regular nonexpansive operators play an important role in fixed point iterations. An example of an asymptotically regular operator is an SQNE one [Ceg12, Theorem 3.4.3]. Opial [Opi67, Theorem 1] proved that any orbit of an AR and NE operator S with $\text{Fix } S \neq \emptyset$ converges weakly to a fixed point of S . It turns out that the same result one can obtain in a more general case. First we recall a notion of the demi-closedness principle.

Definition 7 We say that an operator $U : \mathcal{H} \rightarrow \mathcal{H}$ satisfies the *demi-closedness* (DC) *principle* if

$$(x^k \rightharpoonup x \text{ and } \|Ux^k - x^k\| \rightarrow 0) \implies x \in \text{Fix } S. \quad (12)$$

If (12) holds then we also say that $U - \text{Id}$ is *demi-closed* at 0.

Opial proved that a nonexpansive operator satisfies the DC principle [Opi67, Lemma 2]. Basing on this fact Opial proved in his famous result [Opi67, Theorem 1] that any orbit of a nonexpansive and asymptotically regular operator T having a fixed point converges weakly to $x^* \in \text{Fix } T$. The demi-closedness principle also holds for a subgradient projection P_f , for a continuous convex function $f : \mathcal{H} \rightarrow \mathbb{R}$ with $S(f, 0) := \{x \in \mathcal{H} : f(x) \leq 0\} \neq \emptyset$, which is Lipschitz continuous on bounded subsets (see [Ceg12, Theorem 4.2.7]).

Now we present further properties of a Landweber-type operator. We apply ideas of [Ceg12, Section 2.4] to a Landweber-type operator V . Recall that an operator $S_\tau : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ defined by $S_\tau x := x + \tau(x)(Sx - x)$ is called a *generalized relaxation* of $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$, where $\tau : \mathcal{H}_1 \rightarrow (0, +\infty)$ is called a *step-size function*. If $\tau(x) \geq 1$ for all $x \in \mathcal{H}_1$, then S_τ is called an *extrapolation* of S .

Let $V := \text{Id} + \frac{1}{\|A\|^2} A^*(T - \text{Id})A$ be a Landweber-type operator. Denote

$$U := \text{Id} + A^*(T - \text{Id})A. \quad (13)$$

Then, obviously,

$$Ux - x = \|A\|^2(Vx - x). \quad (14)$$

Let a step-size function $\sigma : \mathcal{H}_1 \rightarrow (0, +\infty)$ be defined by

$$\sigma(x) := \begin{cases} \frac{\|TAx - Ax\|^2}{\|A^*(TAx - Ax)\|^2}, & \text{if } Ax \notin \text{Fix } T, \\ 1, & \text{otherwise.} \end{cases} \quad (15)$$

An operator U_σ defined by $U_\sigma x := x + \sigma(x)A^*(TAx - Ax)$ is a generalized relaxation of U . Denoting

$$\tau(x) = \begin{cases} \|A\|^2\sigma(x), & \text{if } Ax \notin \text{Fix } T, \\ 1, & \text{otherwise.} \end{cases} \quad (16)$$

we obtain

$$V_\tau x = \begin{cases} x + \frac{\|TAx - Ax\|^2}{\|A^*(TAx - Ax)\|^2} A^*(TAx - Ax), & \text{if } Ax \notin \text{Fix } T, \\ x, & \text{otherwise.} \end{cases} \quad (17)$$

By $\|A^*u\| \leq \|A^*\| \cdot \|u\| = \|A\| \cdot \|u\|$, we have $\sigma(x) \geq \|A\|^{-2}$ and $\tau(x) \geq 1$. Therefore, V_τ is an extrapolation of the Landweber-type operator V . Note, however, that we do not need to know the norm of A in order to evaluate $V_\tau x$. In the theorem below we present important properties of V_τ .

Theorem 8 *Let $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator with $\|A\| > 0$, $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ be an α -SQNE operator with $\text{im } A \cap \text{Fix } T \neq \emptyset$, where $\alpha \geq 0$. Further, let an extrapolation V_τ of a Landweber-type operator be defined by (17). Then:*

- (i) $\text{Fix } V_\tau = \text{Fix } V = A^{-1}(\text{Fix } T)$;
- (ii) V_τ is α -SQNE;
- (iii) If $\alpha > 0$ then V_τ is asymptotically regular;
- (iv) If T satisfies the demi-closedness principle then V_τ also satisfies the demi-closedness principle.

Proof. By (4), T is α -SQNE, where $\alpha \geq 0$, if and only if

$$\lambda \langle Tu - u, y - u \rangle \geq \|Tu - u\|^2 \quad (18)$$

for all $u \in \mathcal{H}_2$ and all $y \in \text{Fix } T$, where $\lambda = 2/(\alpha + 1) \in (0, 2]$. Note that $z \in A^{-1}(\text{Fix } T)$ if and only if $Az \in \text{Fix } T$.

(i) The equality $\text{Fix } V_\tau = \text{Fix } V$ is clear, because $\sigma(x) > 0$ for all $x \in \mathcal{H}_1$. Now we prove the second equality.

\supseteq Let $Az \in \text{Fix } T$. Then $Vz = z + (\frac{1}{\|A\|^2} A^*(TAz - Az)) = z$.

\subseteq Let $z \in \text{Fix } V$. Then, of course, $A^*(TAz - Az) = 0$. Let $w \in \mathcal{H}_1$ be such that $Aw \in \text{Fix } T$. By (18), we have

$$\|TAz - Az\|^2 \leq \lambda \langle TAz - Az, Aw - Az \rangle = \lambda \langle A^*(TAz - Az), w - z \rangle = 0,$$

i.e., $Az \in \text{Fix } T$.

(ii) Let $z \in \text{Fix } V_\sigma = \text{Fix } V$. By (i) $Az \in \text{Fix } T$. (14) and (18) yield

$$\begin{aligned} & \lambda \langle Vx - x, z - x \rangle \\ &= \lambda \|A\|^{-2} \langle Ux - x, z - x \rangle = \lambda \|A\|^{-2} \langle A^*(TAx - Ax), z - x \rangle \\ &= \lambda \|A\|^{-2} \langle TAx - Ax, Az - Ax \rangle \geq \|A\|^{-2} \cdot \|TAx - Ax\|^2 \\ &= \|A\|^{-2} \sigma(x) \|A^*(TAx - Ax)\|^2 = \|A\|^{-2} \sigma(x) \|Ux - x\|^2 \\ &= \tau(x) \|Vx - x\|^2, \end{aligned}$$

where $\lambda = 2/(\alpha + 1) \in (0, 2]$, $U(x)$, $\sigma(x)$ and $\tau(x)$ are defined by (13) and (15)–(16), $x \in \mathcal{H}_1$. This yields $\lambda \langle Vx - x, z - x \rangle \geq \tau(x) \|Vx - x\|^2$. Multiplying both sides by $\tau(x)$ we obtain $\lambda \langle V_\tau x - x, z - x \rangle \geq \|V_\tau x - x\|^2$. By (4), the operator V_σ is α -SQNE.

(iii) Follows from (ii) and from the fact that any SQNE operator is AR (see [Ceg12, Theorem 3.4.3]).

(iv) Suppose, that T satisfies the DC principle, i.e., $y^k \rightharpoonup y$ together with $\|Ty^k - y^k\| \rightarrow 0$ implies $y \in \text{Fix } T$. We prove that V_τ also satisfies the demi-closedness principle. Let $x^k \rightharpoonup x$ and $\|V_\tau x^k - x^k\| \rightarrow 0$. Then, obviously, $\|Vx^k - x^k\| \rightarrow 0$, because $\tau(x^k) \geq 1$. Clearly, for any $u \in \mathcal{H}_2$,

$$\lim_k \langle Ax^k - Ax, u \rangle = \lim_k \langle x^k - x, A^*u \rangle = 0,$$

i.e., $Ax^k \rightharpoonup Ax$. Choose an arbitrary $z \in \text{Fix } V_\tau$. By (i), $Az \in \text{Fix } T$. By (18), the Cauchy–Schwarz inequality and the boundedness of x^k , we have

$$\begin{aligned} \|TAx^k - Ax^k\|^2 &\leq \lambda \langle TAx^k - Ax^k, Az - Ax^k \rangle \\ &= \lambda \|A\|^2 \left\langle \frac{1}{\|A\|^2} A^*(TAx^k - Ax^k), z - x^k \right\rangle \\ &\leq \lambda \|A\|^2 \|Vx^k - x^k\| \cdot \|z - x^k\| \rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

Consequently, $\lim_k \|T(Ax^k) - Ax^k\| = 0$. This, together with $Ax^k \rightharpoonup Ax$ and with the assumption that T satisfies the demi-closedness principle, gives $Ax \in \text{Fix } T$, i.e., $x \in A^{-1}(\text{Fix } T) = \text{Fix } V_\tau$ and the proof is completed. ■

By an application of the inequality $\tau(x) \geq 1$, $x \in \mathcal{H}_1$, one can prove that Theorem 8 implies the following result which is a slight generalization of a result due to Wang and Xu [WX11, Lemma 3.1 and Theorem 3.3].

Corollary 9 *Let $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator with $\|A\| > 0$, $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ be an α -SQNE operator with $\text{im } A \cap \text{Fix } T \neq \emptyset$, where $\alpha \geq 0$. Further, let $V := \text{Id} + \frac{1}{\|A\|^2} A^*(T - \text{Id})A$. Then:*

- (i) $\text{Fix } V = A^{-1}(\text{Fix } T)$ and
- (ii) V is α -SQNE.

If, moreover, T satisfies the DC principle, then V also satisfies the DC principle.

A proof of Corollary 9 can be also found in [Ceg14, Lemma 4.1]. One can obtain similar results for a composition of an extrapolation of a Landweber-type operator and an SQNE operator.

Corollary 10 *Let $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ α -SQNE, $U : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ be β -SQNE with $\text{Fix } U \cap A^{-1}(\text{Fix } T) \neq \emptyset$, where $\alpha, \beta > 0$. Further, let $R := U \circ V_\tau$, where V_τ is defined by (17). Then*

- (i) $\text{Fix } R = \text{Fix } U \cap A^{-1}(\text{Fix } T)$;
- (ii) R is γ -SQNE, where $\gamma = (1/\alpha + 1/\beta)^{-1}$;

(iii) R is asymptotically regular.

If, moreover, T and U satisfy the DC principle, then R also satisfies the DC principle.

Proof. By Theorem 8 (i)-(ii) and by the assumption that $\text{Fix } U \cap A^{-1}(\text{Fix } T) \neq \emptyset$, V_τ and U are SQNE operators having a common fixed point. Therefore, (i) follows from [BB96, Proposition 2.10 (i)]. Part (ii) follows now from Theorem 8 (ii), from the assumption that U is β -SQNE and from [Ceg12, Theorem 2.1.48 (ii)]. Part (iii) follows from (i), (ii) and from [Ceg14, Theorem 3.4.3]. Suppose now, that T and U satisfy the DS principle. By Theorem 8, V_τ satisfies the DC principle. Therefore, [Ceg14, Theorem 4.2] yields, that R also satisfies the DC principle. ■

The results presented in Theorem 8 and Corollary 10 can be applied to a proof of weak convergence of sequences generated by the iteration below. Let $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ and $U : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ be QNE operators with $\text{Fix } U \cap A^{-1}(\text{Fix } T) \neq \emptyset$, $T_k := T_{\lambda_k}$ and $U_k := U_{\mu_k}$ be their relaxations, where the relaxation parameters $\lambda_k, \mu_k \in [\varepsilon, 1 - \varepsilon]$ for some small $\varepsilon > 0$. Further, let

$$V_k := \begin{cases} x + \frac{\|T_k Ax - Ax\|^2}{\|A^*(T_k Ax - Ax)\|^2} A^*(T_k Ax - Ax), & \text{if } Ax \notin \text{Fix } T, \\ x, & \text{otherwise.} \end{cases} \quad (19)$$

be an extrapolation of a Landweber-type operator related to T_k . Consider the following iterative process

$$x^{k+1} = U_k V_k x^k, \quad (20)$$

where $x^0 \in \mathcal{H}_1$.

Corollary 11 *Let $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ and $U : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ be QNE operators with $\text{Fix } U \cap A^{-1}(\text{Fix } T) \neq \emptyset$ and satisfying the demi-closedness principle. Then for any starting point $x^0 \in \mathcal{H}_1$ the sequence $\{x^k\}_{k=0}^\infty$ generated by (20) converges weakly to a point $x^* \in \text{Fix } U \cap A^{-1}(\text{Fix } T)$.*

Proof. It is easily seen that U_k is α_k -SQNE and T_k is β_k -SQNE, where $\alpha_k = \frac{1-\lambda_k}{\lambda_k}$, $\beta_k = \frac{1-\mu_k}{\mu_k}$ and $\alpha_k, \beta_k \in [\varepsilon, \frac{1}{\varepsilon}]$. By Corollary 10 (i)-(ii), $\text{Fix}(U_k \circ V_k) = \text{Fix } U \cap A^{-1}(\text{Fix } T)$ and $U_k V_k$ is γ_k -SQNE with $\gamma_k = (\frac{1}{\alpha_k} + \frac{1}{\beta_k})^{-1} \in [\frac{\varepsilon}{2}, \frac{1}{2\varepsilon}]$. Therefore, the Corollary follows from a more general result [Ceg14, Theorem 5.1]. ■

Remark 12 A special case of iteration (20) was recently presented by López *et al.* [LMWX12], where T_k and U_k are subgradient projections related to subdifferentiable convex functions c and q , respectively, with $C := \{x \in \mathcal{H}_1 : c(x) \leq 0\}$ and $Q := \{y \in \mathcal{H}_2 : q(y) \leq 0\}$. López *et al.* [LMWX12, Theorem 4.3] proved the weak convergence of x^k to a solution of the SFP (8). Very recently, Cui and Wang [CW14] studied the split fixed point problem: find $x \in \text{Fix } V$ with $Ax \in \text{Fix } T$, where $V : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ and $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ are β -demicontractive operators. Recall that an operator $S : \mathcal{H} \rightarrow \mathcal{H}$ having a fixed point is called β -contractive if

$$\|Tx - z\|^2 \leq \|x - z\|^2 + \beta \|Tx - x\|^2,$$

for all $x \in \mathcal{H}_1$ and $z \in \text{Fix } T$ (cf. [Mou10]). Cui and Wang applied a generalized relaxation of the operator U defined by (13) for solving this problem. Roughly spoken, Cui and Wang [CW14, Theorem 3.3] observed that any orbit of the operator $V_\lambda \circ U_\rho$, where the step-size $\rho(x) = \frac{1-\beta}{2}\sigma(x)$ and the relaxation parameter $\lambda \in (0, 1-\beta)$, converges weakly to a solution of the problem. Their result is interesting, however, an application of demicontractive operators seems to be artificial, because $T_{\frac{1-\beta}{2}}$ is a cutter, consequently U_ρ is a cutter, and V_λ is strongly quasi-nonexpansive.

Before we present the next property of a Landweber-type operator, we recall a notion of an approximately shrinking operator.

Definition 13 ([CZ14, Definition 3.1]) Let $U : \mathcal{H} \rightarrow \mathcal{H}$ be quasi-nonexpansive. We say that U is *approximately shrinking* (AS) if for any bounded subset $D \subseteq \mathcal{H}$ and for any $\eta > 0$ there is $\gamma > 0$ such that for any $x \in D$ it holds

$$\|Ux - x\| < \gamma \implies d(x, \text{Fix } U) < \eta. \quad (21)$$

Important examples of AS operators are the metric projection P_C onto a closed convex subset $C \subseteq \mathcal{H}$ and the subgradient projection P_f for a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ [CZ13, Lemma 24].

Theorem 14 Let $T : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a QNE operator with $\text{im } A \cap \text{Fix } T \neq \emptyset$. Further, let $V := \text{Id} + \frac{1}{\|A\|^2} A^*(T - \text{Id})A$ be a Landweber-type operator and V_τ be its extrapolation, where τ is given by (16). If T is approximately shrinking then V and V_τ are also approximately shrinking.

The theorem will be proved elsewhere. It turns out that the properties presented in Theorems 8 and 14 and in Corollary 10 can be applied to a proof of the strong convergence of sequences generated by a hybrid steepest descent method with an application of a Landweber-type operators to a solution of a variational inequality (see [CZ13] and [CZ14] for the details).

4 Examples

Example 15 Let $a \in \mathcal{H}$ with $\|a\| > 0$, $A : \mathcal{H} \rightarrow \mathbb{R}$ be defined by $Ax = \langle a, x \rangle$. Then $\|A\| = \|a\|$. If $Q := (-\infty, \beta]$, where $\beta \in \mathbb{R}$, then the Landweber operator V and its extrapolation V_τ with a step-size function τ defined by (16) coincide and

$$Vx = V_\tau x = P_C x = x + \frac{(\langle a, x \rangle - \beta)_+}{\|a\|^2} a,$$

where $C = H(a, \beta)_- := \{u \in \mathcal{H} : \langle a, u \rangle \leq \beta\}$ and $\alpha_+ := \max\{\alpha, 0\}$.

Example 16 Let $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator with $\|A\| > 0$. If $Q = \{b\}$, where $b \in \mathcal{H}_2$, then the SFP is to find a solution of a linear system $Ax = b$. The Landweber operator V related to this problem and its extrapolation V_τ with a step-size function τ defined by (16) have the forms

$$Vx = x - \frac{1}{\|A\|^2} A^*(Ax - b)$$

and

$$V_\tau x = x - \frac{\|Ax - b\|^2}{\|A^*(Ax - b)\|^2} A^*(Ax - b).$$

By Theorem 8 and Corollary 9, V and V_τ satisfy the DC principle.

Example 17 Let A be an $m \times n$ real matrix representing a linear operator and $b \in \mathbb{R}^m$. Suppose without loss of generality that the rows a_i of A , $i = 1, 2, \dots, m$, are nonzero vectors. If $Q := \{u \in \mathbb{R}^m : u \leq b\} = b - \mathbb{R}_+^m$, then the SFP is to find a solution of a system of linear inequalities $Ax \leq b$. The Landweber operator V related to this problem and its extrapolation V_τ with a step-size function τ defined by (16) have the forms

$$Vx = x - \frac{1}{\lambda_{\max}(A^T A)} A^T (Ax - b)_+$$

and

$$V_\tau x = x - \frac{\|(Ax - b)_+\|^2}{\|A^*(Ax - b)_+\|^2} A^T (Ax - b)_+,$$

where A^T denotes the transposed matrix and $\lambda_{\max}(A^T A)$ denotes the maximal eigenvalue of $A^T A$. By Theorem 8 and Corollary 9, V and V_τ satisfy the DC principle. Moreover, by Theorem 14, V and V_τ are approximately shrinking.

Example 18 Let $f : \mathcal{H}_2 \rightarrow \mathbb{R}$ be a continuous convex function with $S(f, 0) := \{y \in \mathcal{H}_2 : f(y) \leq 0\} \neq \emptyset$. Define a subgradient projection $P_f : \mathcal{H}_2 \rightarrow \mathcal{H}_2$, related to f by

$$P_f(y) = \begin{cases} y - \frac{f(y)_+}{\|g_f(y)\|^2} g_f(y), & \text{if } f(y) > 0, \\ y, & \text{otherwise,} \end{cases}$$

where $g_f(y)$ denotes a subgradient of f at $y \in \mathcal{H}_2$. Suppose that f is Lipschitz continuous on bounded subsets. Let $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator with $\|A\| > 0$, $\lambda \in (0, 2]$ and $T := P_{f, \lambda}$. Then T is α -SQNE with $\alpha = \frac{2-\lambda}{\lambda}$. A Landweber operator V related to this problem and its extrapolation V_τ with a step-size function τ defined by (16) have the forms

$$Vx = \begin{cases} x - \frac{\lambda f(Ax)_+}{(\|A\| \cdot \|g_f(Ax)\|)^2} A^* g_f(Ax), & \text{if } f(Ax) > 0, \\ x, & \text{otherwise} \end{cases}$$

and

$$V_\tau x = \begin{cases} x - \frac{\lambda f(Ax)_+}{\|A^* g_f(Ax)\|^2} A^* g_f(Ax), & \text{if } f(Ax) > 0, \\ x, & \text{otherwise.} \end{cases}$$

Because P_f satisfies the DC principle (see [Ceg12, Theorem 4.2.7]), Theorem 8 and Corollary 9 yield, that V and V_τ satisfy the DC principle. Moreover, if \mathcal{H}_2 is finite-dimensional, then P_f is approximately shrinking (see [CZ13, Lemma 24]). Therefore, Theorem 14 yields that in this case V and V_τ are also approximately shrinking.

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